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REGULARIZING EFFECTS FOR U SUB T + A(Psi(U))=0 IN L1 VECTOR SPA--ETC(U)

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(9) MRC Technical Summary Report #2187

REGULARIZING EFFECTS FOR

(10) $u_t + A(u) = 0$ IN L^1

Michael G. Crandall and Michel Pierre

(6) Regularizing Effects for
 $u_{sub} + A(\psi(u)) = 0$
in L^1 vector space

(14) MRC-TSR-2187

Mathematics Research Center
University of Wisconsin-Madison
610 Walnut Street
Madison, Wisconsin 53706

(15) DAAG29-80-C-0041
NSF-MCS79-27062

(11) March 1981

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Received January 6, 1981

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Approved for public release
Distribution unlimited

Sponsored by
U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

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REGULARIZING EFFECTS FOR $u_t + A\varphi(u) = 0$ IN L^1 .

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ABSTRACT

Various initial-boundary value problems and Cauchy problems can be written in the form $\frac{du}{dt} + A\varphi(u) = 0$ where $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing and A is the linear generator of strongly continuous nonexpansive semigroup e^{-tA} in an L^1 space. For example, if $A = -\Delta$ (subject, perhaps, to suitable boundary conditions) we obtain equations arising in flow in a porous medium or plasma physics (depending on the choice of φ) while if $A = \frac{\partial}{\partial x}$ acting in $L^1(\mathbb{R})$ we have a scalar conservation law. In this paper we show that if $M, m > 0$ and $m\varphi'^2 \leq v\varphi\varphi'' \leq M\varphi'^2$, where $v \in \{1, -1\}$, then (roughly speaking), the norm of tdu/dt may be estimated in terms of the initial data u_0 in L^1 . Such estimates give information about the regularity of solutions, asymptotic behaviour, etc., in applications.

Side issues, such as the introduction of sufficiently regular approximate problems on which estimates can be made and the assignment of a precise meaning to the operator " $A\varphi$ ", are also dealt with. These considerations are of independent interest.

AMS(MOS) Subject Classification: 35K15, 35K55

Key Words: regularizing effect, porous flow equations, conservation laws, nonlinear semigroups

Work Unit No. 1 - Applied Analysis

Sponsored by the United States Army under Contract No. DAAG29-80-C-41. This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062.

L^1 VECTOR SPACE

$du/dt + A(\psi(u)) = 0$
 $\psi: \mathbb{R} \rightarrow \mathbb{R}$ APPROACHES \mathbb{R}

e^{-tA} \rightarrow TO THE $-tA$ POWER

\rightarrow next page

SIGNIFICANCE AND EXPLANATION

Many models of interesting phenomena yield equations for the evolution of a system of the ^{th.s} abstract form $u' + A\phi(u) = 0$ where ϕ ^{psi} is a nonlinear nondecreasing function and A is an "operator". E.g., A may be the $\partial^2/\partial x^2$ Laplacian (perhaps under boundary conditions) or A may be $\frac{\partial}{\partial x}$, while ϕ _{L psi} may be a power law, $\phi(x) = |x|^{\alpha-1}x$. Models like this occur in porous flow, plasmas and conservation laws. In this work it is shown that a broad class of such problems are solvable by the nonlinear semigroup theory. The main point, however, is a "regularizing" effect which estimates the speed of the system at time $t > 0$ by the integral of the initial data. This has consequences for the regularity of the solutions of concrete problems and their asymptotic behaviour.

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REGULARIZING EFFECTS FOR $u_t + A\varphi(u) = 0$ IN L^1 .

Michael G. Crandall and Michel Pierre

Introduction.

When applied to a solution u of the equation

$$(1) \quad u_t - \Delta\varphi(u) = 0 \quad \text{in} \quad (0, \infty) \times \mathbb{R}^N$$

one of the main results of this paper implies that

$$(2) \quad \int_{\mathbb{R}^N} |u_t(t, x)| dx \leq \frac{C}{t} \int_{\mathbb{R}^N} |u(0, x)| dx$$

provided φ is nondecreasing, $\varphi(0) = 0$ and has the property

$$(3) \quad 0 < m \leq v \frac{\varphi(r) \varphi''(r)}{(\varphi'(r))^2} \leq M \quad \text{a.e. } r \in \mathbb{R} \quad \text{for } v = 1 \text{ or } v = -1.$$

Indeed, when (3) holds so does (2) and C depends only on the structure constants m

and M of (3). Note that the initial data $u(0, x)$ need only belong to $L^1(\mathbb{R}^N)$.

The validity of the " L^1 -regularizing" inequality (2) depends strongly on the properties of the operator " $-\Delta\varphi$ " in the space $L^1(\mathbb{R}^N)$. These properties are in fact enjoyed by a large

class of operators of the form $A\varphi$ where φ is as above and A is a linear operator in an L^1 space. Indeed, it is enough that $-A$ be the infinitesimal generator of a strongly continuous nonexpansive semigroup e^{-tA} in L^1 such that $0 < u_0 < 1$ a.e. implies

$0 < e^{-tA} u_0 < 1$ a.e. (i.e., e^{-tA} is submarkovian). Thus the results apply to (1) set

in a bounded domain with linear homogeneous boundary conditions of Dirichlet or Neumann

type imposed on $\varphi(u)$. Similarly, $-\Delta$ can be replaced by more general elliptic operators

and we can, for example, also exhibit the conservation law

$$(4) \quad u_t - \varphi(u)_x = 0 \quad \text{on} \quad (0, \infty) \times \mathbb{R}$$

as an example of the theory developed here.

The estimate (2) is already known if $\varphi(r) = r$. In this event (1) is the linear heat equation and (2) says that $-\Delta$ generates an analytic semigroup in $L^1(\mathbb{R}^N)$, which is

obvious from the solution formula. There has not been much success in developing a general

nonlinear analogue of the linear idea of an analytic semigroup and only a few nonlinear

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062.

results with estimates like (2) have been found. We refer to [7] for more comments in this direction as well as to [4] where a large class of homogeneous nonlinearities are exhibited which permit estimates like (2). The main contribution of this paper is the introduction of interesting new classes of such nonlinear examples.

If $\varphi(r) = |r|^\alpha \text{sign} r$ with $\alpha > 0$, then $\varphi(r)\varphi''(r)/(\varphi'(r))^2 = (\alpha-1)/\alpha$ and (3) holds with $m = [\alpha-1]/\alpha = M$ and $v = \text{sign}(\alpha-1)$ if $\alpha \neq 1$. In this case (1) is covered by the results of [7]. Note that we exclude $\alpha = 1$ here. As mentioned in [7], this is not surprising since the proof of our results also applies to (4) and no estimate like (2) holds if $\varphi(r) = r$ in (4).

We also show in this paper that nonnegative solutions of

$$(5) \quad \frac{du}{dt} + A\varphi(u) = 0,$$

(which is given a precise sense in the text) satisfy a pointwise estimate

$$(6) \quad u_t \geq -C \frac{u}{t}$$

for the class of operators $A\varphi$ where A is as above and the nondecreasing function φ satisfies $\varphi(0) = 0$ and

$$(7) \quad 0 < m \leq \frac{\varphi(r)\varphi''(r)}{(\varphi'(r))^2} \quad \text{a.e. } r > 0.$$

It was previously observed by L. C. Evans and one of the authors that (7) implies (6) for nonnegative solutions of (1). (Pointwise estimates like (6) are enjoyed only by nonnegative solutions.) For $\varphi(r) = r^\alpha$, $\alpha > 0$, this was first shown in the case of (1) by Aronson and Benilan [1] while [7] covers a general class of homogeneous nonlinearities. The paper [8] covers (1) for a quite general class of nonlinearities (considerably more general than [7]), but this result requires extensive exploitation of special properties of the Laplace operator. Here our result is more abstract, in the spirit of [7].

The first section is devoted to the abstract results. As usual, the problem of defining " $A\varphi$ " (and hence (5)) in a precise sense must be disposed of. Similarly, the appropriate meaning must be given to (2), its abstract analogue, and (6). These matters and the approximations introduced in the proofs of the main results are of substantial independent interest. Several proofs of results used in the sequel are collected in the Appendix.

Section 1.

Throughout this section Ω denotes a σ -finite measure space with the measure denoted by "meas". The norm of $L^p(\Omega)$ is denoted by $\| \cdot \|_p$. The integral of $f \in L^1(\Omega)$ over a measurable $O \subset \Omega$ is written either as $\int_O f$ or $\int_O f dx$.

Recall that a (possibly nonlinear) mapping $A: D(A) \subset X \rightarrow X$ in a Banach space X is accretive if for each $\lambda > 0$ $(I + \lambda A)^{-1}$ is a nonexpansive mapping of $R(I + \lambda A)$ (the range of $I + \lambda A$) into X . If A is accretive and $R(I + \lambda A) = X$ for $\lambda > 0$ (equivalently, $R(I + A) = X$), then A is m-accretive. If A is linear and densely defined, then A is m-accretive if and only if $-A$ is the infinitesimal generator of a (linear) strongly continuous nonexpansive semigroup e^{-tA} on X . More generally, if A is accretive and $R(I + \lambda A) \supseteq \overline{D(A)}$ for $\lambda > 0$ it determines a (in general, nonlinear) strongly continuous nonexpansive semigroup e^{-tA} on $\overline{D(A)}$. (We use the notation e^{-tA} in the linear and nonlinear cases.) See, e.g. [2], [6], [9].

We assume a densely defined linear operator $A: D(A) \subset L^1(\Omega) + L^1(\Omega)$ is given which satisfies

$$(A1) \quad A \text{ is m-accretive in } L^1(\Omega)$$

and

$$(A2) \quad \begin{cases} \text{If } \lambda > 0, f \in L^1(\Omega), a, b \in \mathbb{R} \text{ and } a < f < b \text{ a.e.,} \\ \text{then } a < (I + \lambda A)^{-1} f < b \text{ a.e.} \end{cases}$$

Since A is linear, densely defined and m-accretive, (A2) is equivalent to

$$0 < f < 1 \implies 0 < e^{-tA} f < 1. \quad (\text{Actually, (A1) and (A2) imply } D(A) \text{ is dense ([10]).})$$

It was proved in [5] that for linear m-accretive A 's as above, (A2) is equivalent to

$$(A3) \quad \begin{cases} \text{If } \beta \text{ is a maximal monotone graph in } \mathbb{R} \times \mathbb{R} \text{ with } 0 \in \beta(0), u \in D(A), \\ Au \in L^p(\Omega), 1 < p < \infty, v \in L^{p/(p-1)}(\Omega), v(x) \in \beta(u(x)) \text{ a.e. then} \\ \int_{\Omega} v(x) Au(x) dx > 0. \end{cases}$$

The proper interpretation of " $\lambda\psi$ " is discussed next. Set

$$P_0 = \{ \psi: \mathbb{R} \rightarrow \mathbb{R}; \psi \text{ is continuous, nondecreasing and } \psi(0) = 0 \}.$$

For any $\psi \in P_0$ and $B: D(B) \subseteq L^1(\Omega) + L^1(\Omega)$ the operator $B\psi$ in $L^1(\Omega)$ is first defined in the obvious way:

$$(1.1) \quad \begin{cases} D(B\varphi) = \{u \in L^1(\Omega); \varphi(u) \in D(B)\} \\ \forall u \in D(B\varphi), \quad B\varphi(u) = B(\varphi(u)). \end{cases}$$

The proposition below summarizes some results which follow easily from the results and arguments of, e.g., [5].

Proposition 1. Let A be linear, densely defined and satisfy (A1), (A2). Let $\varphi \in P_0$.

Then:

- (i) $A\varphi$ is accretive in $L^1(\Omega)$.
- (ii) For each $\varepsilon > 0$ and $\lambda > 0$, $\varepsilon I + A(I + \lambda A)^{-1}$ satisfies (A1), (A2).
- (iii) For each $\varepsilon > 0$, $(\varepsilon I + A)\varphi$ is m -accretive in $L^1(\Omega)$.
- (iv) For $\lambda > 0$, $(I + \lambda A\varphi)^{-1}$ is an order-preserving nonexpansive mapping of $R(I + \lambda A\varphi)$ into $L^1(\Omega)$. Moreover, $f \in R(I + \lambda A\varphi)$ $a, b \in R$ and $a \leq f \leq b$ a.e. implies $a \leq (I + \lambda A\varphi)^{-1}f \leq b$ a.e.
- (v) $\|u\|_p \leq \|(I + \lambda A\varphi)(u)\|_p$ for $u \in D(A\varphi)$, $1 \leq p \leq \infty$.

The main omission of Proposition 1 is the assertion that $A\varphi$ is m -accretive. In general this fails even if A satisfies (A1), (A2). However, the pair (A, φ) typically determines an m -accretive operator A_φ which extends $A\varphi$ and (A, φ) always determines an accretive operator A_φ for which $R(I + \lambda A_\varphi) \supset L^1(\Omega)^+$ as is stated in the next proposition.

Proposition 2. Let A be linear, densely defined and satisfy (A1), (A2). Let $\varphi \in P_0$ and assume at least one of the conditions:

- (i) φ is strictly increasing,
- (ii) $\exists r_0 > 0$, K such that $|\varphi(r)| \leq K|r|$ for $|r| \leq r_0$,
or
- (iii) $\text{meas}(\Omega) < \infty$.

Then there is an m -accretive operator A_φ in $L^1(\Omega)$ which extends $A\varphi$ such that for every $\lambda > 0$ and $f \in L^1(\Omega)$

$$(1.2) \quad \lim_{\varepsilon \rightarrow 0} (I + \lambda(\varepsilon I + A)\varphi)^{-1}f = (I + \lambda A_\varphi)^{-1}f.$$

Moreover, for every $\varphi \in P_0$ there exists an accretive operator A_φ in $L^1(\Omega)$ which extends $A\varphi$ such that (1.2) holds for every $\lambda > 0$ and $f \in L^1(\Omega)^+ = \{f \in L^1(\Omega): f \geq 0\}$.

Proposition 2 is tangential to our main concerns and is discussed and proved in the Appendix.

Each φ we deal with will allow the application of Proposition 2, and we take A_φ to be the correct interpretation of $A\varphi$ in (5). Solutions of (5) are then understood in the sense of nonlinear semigroup theory - i.e. $u(t) = e^{-tA_\varphi} u(0)$. An important fact for our presentation is the:

Convergence Theorem: Let G_n , $n = 1, 2, \dots, \infty$ be a sequence of accretive operators in $L^1(\Omega)$ such that $\overline{D(G_n)} \supset D(G_\infty)$ and $R(I + \lambda G_n) \supset \overline{D(G_n)}$ for $n = 1, 2, \dots, \infty$ and $\lambda > 0$. Assume

$$\lim_{n \rightarrow \infty} (I + \lambda G_n)^{-1} f = (I + \lambda G_\infty)^{-1} f$$

for $f \in D(G_\infty)$ and $\lambda > 0$. Then whenever $f_n \in \overline{D(G_n)}$ and $f_n \rightarrow f_\infty \in \overline{D(G_\infty)}$ we have

$$\lim_{n \rightarrow \infty} e^{-tG_n} f_n = e^{-tG_\infty} f_\infty$$

uniformly for bounded $t > 0$. (All convergences are in $L^1(\Omega)$.)

This theorem is a special case of known results (see, e.g., [6] for references). It follows from Proposition 1 and the convergence theorem that $\lim_{\varepsilon \rightarrow 0} e^{-t(\varepsilon I + A)\varphi} u_\varepsilon = e^{-tA_\varphi} u$ uniformly for bounded $t > 0$ whenever $u_\varepsilon \in D((\varepsilon I + A)\varphi)$ converges to $u \in \overline{D(A_\varphi)}$.

Our main goal is to estimate the speed of the semigroup e^{-tA_φ} generated by $-A_\varphi$ under suitable assumptions on φ . We will prove:

Theorem 3. Let A be linear, densely defined, satisfy (A1), (A2) and $\varphi \in P_0$. Assume

$$(1.3) \quad \varphi \in C^1(\mathbb{R} \setminus \{0\}), \quad \varphi' \text{ is locally Lipschitz on } \mathbb{R} \setminus \{0\},$$

and

$$(1.4) \quad \text{There exists } m, M > 0 \text{ and } v \in [-1, 1] \text{ such that}$$

$$m(\varphi'(x))^2 \leq v\varphi(x)\varphi''(x) \leq M(\varphi'(x))^2 \quad \text{a.e. } x \in \mathbb{R}.$$

Then φ satisfies either (i) or (ii) of Proposition 2 and for $S(t) = e^{-tA_\varphi}$,

$$u_0 \in \overline{D(A_\varphi)},$$

$$(1.5) \quad \lim_{h \neq 0} \frac{\|S(t+h)u_0 - S(t)u_0\|_1}{h} < \frac{C}{t} \|u_0\|_1$$

where $C = 2(M+1)(m+2M)/m^2$.

Remarks:

(a) The assumption (1.4) is a natural generalization of the condition $\varphi\varphi''/(\varphi')^2 = C \neq 0$ which is the homogeneous case treated in [7]. Note that $v = 1$ and $v = -1$ correspond to quite different behaviours of φ . For instance, if $v = 1$ then φ is convex on $[0, \infty)$ while if $v = -1$ it is concave.

One can easily see that for $\varphi \in P_0$, (1.3) and (1.4) are equivalent to

$$(1.6) \quad \begin{cases} \varphi \in C^1(\mathbb{R} \setminus \{0\}), \varphi/\varphi' \text{ is Lipschitz continuous on } \mathbb{R} \text{ and} \\ v-m > v(\frac{\varphi}{\varphi'})' > v-M \\ \text{(where } \varphi/\varphi' \text{ is understood to vanish if } \varphi(r) = \varphi'(r) = 0 \text{ or } r = 0), \end{cases}$$

or

$$(1.7) \quad \begin{cases} r + \frac{v}{1-vm} |\varphi(r)|^{1-vm} (\log|\varphi(r)| \text{ if } vm \neq 1) \text{ is convex and} \\ r + \frac{v}{1-vM} |\varphi(r)|^{1-vM} (\log|\varphi(r)| \text{ if } vM \neq 1) \\ \text{is concave on each of } (-\infty, 0) \text{ and } (0, \infty). \end{cases}$$

Note that $v = 1$ implies $m < 1$. Also note that if $v = 1$, the convexity implies $|\varphi(r)| \leq K|r|$, $K = \max(\varphi'(r_0+), \varphi'(r_0-))$ on $|r| < r_0$ so Proposition 2(ii) holds, while if $v = -1$ either $\varphi \equiv 0$ on $[0, \infty)$ or φ is strictly increasing by (1.6) and Proposition 2(i) holds.

(B) It would be interesting to know if the existence of the upper bound M in (1.4) is necessary to have an estimate like (1.5). Our next result shows one needs only m if the initial data is nonnegative and $v = 1$.

Theorem 4. Let A be linear, densely defined, satisfy (A1), (A2) and $\varphi \in P_0$. Assume $m > 0$, $m \neq 1$, $v \in \{-1, 1\}$ and

$$(1.8) \quad r + \frac{v}{1-vm} \varphi(r)^{1-vm} \text{ is convex on } (0, \infty).$$

Let A_φ be as in Proposition 2 and $S(t) = e^{-tA_\varphi}$. Then for $u_0 \geq 0$, $u_0 \in \overline{D(A_\varphi)}$

$$(1.9) \quad t \mapsto vt^{v/m} \varphi(S(t)u_0(x)) \text{ is nondecreasing a.e. } x \in \Omega.$$

If also $v = 1$ (so $m < 1$), then

$$(1.10) \quad \lim_{h \rightarrow 0} \frac{S(t+h)u_0 - S(t)u_0}{h} \leq \frac{2(1-m)}{mt} \|u_0\|_1.$$

Remarks: Notice that (1.9) is a weak formulation of

$$vt \frac{du}{dt} \geq -\frac{1}{m} \frac{\varphi(u)}{\varphi'(u)}$$

where $u = S(t)u_0$. If $v = 1$, then $\varphi(u)/\varphi'(u) \leq (1-m)u$, so we obtain

$$t \frac{du}{dt} \geq -\frac{(1-m)}{m} u. \text{ This means } t \mapsto t^{\frac{1-m}{m}} S(t)u_0 \text{ is nondecreasing, which may be deduced from (1.9) directly when } v = 1.$$

We begin the proofs of Theorems 3 and 4. While the formal manipulations which are the basis of the main estimates are quite straightforward, there are considerable difficulties concerning regularity to be overcome. We use a four-layered approximation process to dispose of these difficulties. One has been introduced already, namely the approximation of A_φ by $(\varepsilon I + A)\varphi$. To this we add the regularization of A itself by its Yosida approximation $A_\lambda = \lambda^{-1}(I - (I + \lambda A)^{-1}) = A(I + \lambda A)^{-1}$ and, in turn, the replacement of φ by its Yosida approximation $\varphi_\alpha = \alpha^{-1}(I - (I + \alpha\varphi)^{-1})$. A fourth approximation process is introduced later. We recall that A_λ is m -accretive, defined on all of $L^1(\Omega)$ and bounded. Moreover, by Proposition 1, $\|(I + \lambda A)^{-1}f\|_p \leq \|f\|_p$ for $f \in L^p(\Omega) \cap L^1(\Omega)$, $1 \leq p < \infty$ and $\lambda > 0$. Thus $A_\lambda: L^1(\Omega) \cap L^p(\Omega) \rightarrow L^1(\Omega) \cap L^p(\Omega)$ and A_λ on this domain is accretive and Lipschitz continuous in the $L^p(\Omega)$ norm. The next lemma handles the problem of passing to the limit in the approximation of φ by φ_α as $\alpha \rightarrow 0$.

Lemma 5. Let $\varphi \in P_0$, $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$, $\varphi(u_0) \in L^1(\Omega)$. Let $\varepsilon, \lambda, \alpha > 0$, and $B = \varepsilon I + A_\lambda$. Then the problems

$$(1.11) \quad \frac{du}{dt} + B\varphi_\alpha(u_\alpha) = 0, \quad u_\alpha(0) = u_0,$$

and

$$(1.12) \quad \frac{du}{dt} + B\varphi(u) = 0, \quad u(0) = u_0,$$

have unique solutions $u_\alpha, u \in W^{1,\infty}([0,\infty); L^1(\Omega))$.

Moreover

$$(1.13) \quad \begin{cases} \lim_{\alpha \rightarrow 0} u_\alpha = u & \text{in } C([0,T]; L^1(\Omega)) \\ \lim_{\alpha \rightarrow 0} \frac{du_\alpha}{dt} = \frac{du}{dt} & \text{in } L^1(0,T; L^1(\Omega)) \end{cases}$$

for every $T > 0$.

Proof of Lemma 5. By Proposition 1, $B\varphi_\alpha$ and $B\varphi$ are m -accretive in $L^1(\Omega)$.

Moreover $B\varphi_\alpha \rightarrow B\varphi$ as $\alpha \rightarrow 0$ in the sense $(I + \lambda B\varphi_\alpha)^{-1}f \rightarrow (I + \lambda B\varphi)^{-1}f$ for $f \in L^1(\Omega)$,

$\lambda > 0$. Indeed, if $f \in L^1(\Omega)$ and

$$v_\alpha + \lambda B\varphi_\alpha(v_\alpha) = f, \quad v + \lambda B\varphi(v) = f$$

we also have $v_\alpha - v + \lambda B\varphi_\alpha(v_\alpha) - \lambda B\varphi(v) = \lambda B(\varphi(v) - \varphi_\alpha(v))$. Since $B\varphi_\alpha$ is accretive this yields

$$\|v_\alpha - v\|_1 \leq \lambda \|B\| \|\varphi(v) - \varphi_\alpha(v)\|_1$$

and the right-hand side tends to 0 as $\alpha \rightarrow 0$ because $\varphi(v) \in L^1(\Omega)$, $\|\varphi_\alpha(v)\|_1 \leq \|\varphi(v)\|_1$

and $\varphi_\alpha(v) \rightarrow \varphi(v)$ a.e. by standard properties of the Yosida approximation φ_α . Let

$T_\alpha(t) = e^{-tB\varphi_\alpha}$ and $T(t) = e^{-tB\varphi}$. Since $B\varphi_\alpha \rightarrow B\varphi$, $u_\alpha(t) = T_\alpha(t)u_0 + \int_0^t T_\alpha(t-s)f(s)ds \rightarrow u(t)$ in $L^1(\Omega)$ as $\alpha \rightarrow 0$ uniformly for bounded $t \geq 0$. Now $B\varphi_\alpha$ is Lipschitz continuous so $u_\alpha \in C^1([0,\infty); L^1(\Omega))$ and

$$\left\| \frac{du_\alpha}{dt}(t) \right\|_1 \leq \|B\varphi_\alpha(u_0)\|_1 \quad \text{for } t \geq 0$$

by the accretivity of $B\varphi_\alpha$. As $\|\varphi_\alpha(u_0)\|_1 \leq \|\varphi(u_0)\|_1$, $B\varphi_\alpha(u_0)$ is bounded in $L^1(\Omega)$

independently of $\alpha > 0$ and hence so are du_α/dt and $\varphi_\alpha(u_\alpha) = -B^{-1}(\frac{du_\alpha}{dt})$ bounded in $L^\infty(0,\infty; L^1(\Omega))$. Moreover

$$(1.14) \quad \int_0^T \int_\Omega \varphi_\alpha(u_\alpha) dx dt = \int_\Omega B^{-1}(u_0 - u_\alpha(T)) dx.$$

Since B enjoys the property (A2) together with A , $\|u_\alpha(t)\|_\infty \leq \|u_0\|_\infty$. It follows that $\varphi_\alpha(u_\alpha)$ is bounded in $L^\infty(\Omega)$ uniformly in $\alpha, t \geq 0$ and then, by interpolation, in every $L^p(\Omega)$. We conclude that $\frac{du_\alpha}{dt}$ is bounded in $L^\infty(0,\infty; L^2(\Omega))$. This together with

$u_\alpha \rightarrow u$ in $C([0,\infty); L^1(\Omega))$, shows $\frac{du}{dt} \in L^\infty(0,\infty; L^2(\Omega))$ and $du_\alpha/dt \rightarrow du/dt$ weakly in $L^2(0,T; L^2(\Omega))$ for each $T > 0$. From $\varphi_\alpha(u_\alpha) = B^{-1}(-\frac{du_\alpha}{dt})$ and the boundedness of B^{-1} in $L^2(\Omega)$ it then follows that $\varphi_\alpha(u_\alpha) \rightarrow -B^{-1}(\frac{du}{dt})$ weakly in $L^2(0,T; L^2(\Omega))$. On

the other hand, $\varphi_\alpha(u_\alpha) \rightarrow \varphi(u)$ in measure and so $\varphi(u) = -B^{-1}(\frac{du}{dt})$, which establishes (1.12) and its consequence

$$(1.15) \quad \int_0^T \int_\Omega \varphi(u) dx dt = \int_\Omega B^{-1}(u_0 - u(T)).$$

Assume now $u_0 > 0$ so $u_\alpha, u > 0$. By (1.14), (1.15) and $u_\alpha \rightarrow u$ in $C([0, \infty); L^1(\Omega))$ we conclude $\varphi_\alpha(u_\alpha) \rightarrow \varphi(u)$ in measure and

$$\int_0^T \int_\Omega \varphi_\alpha(u_\alpha) dx dt \rightarrow \int_0^T \int_\Omega \varphi(u) dx dt.$$

Since $\varphi_\alpha(u_\alpha), \varphi(u) > 0$ this implies $\varphi_\alpha(u_\alpha) \rightarrow \varphi(u)$ in $L^1(0, T; L^1(\Omega))$. If u_0 is not of fixed sign we may estimate $\varphi_\alpha(u_\alpha)$ by $\varphi_\alpha(v_\alpha) \leq \varphi_\alpha(u_\alpha) \leq \varphi_\alpha(w_\alpha)$ where $w_\alpha = T_\alpha(t)u_0^+$, $v_\alpha = T_\alpha(t)(-u_0^-)$. Since $\varphi_\alpha(v_\alpha)$ and $\varphi_\alpha(w_\alpha)$ converge in $L^1(0, T; L^1(\Omega))$ and $\varphi_\alpha(u_\alpha)$ converges in measure, $\varphi_\alpha(u_\alpha)$ converges in L^1 . By the continuity of B

$$\frac{du}{dt} = -B\varphi_\alpha(u_\alpha) \rightarrow \frac{du}{dt} \text{ in } L^1(0, T; L^1(\Omega)).$$

This completes the proof of the Lemma.

The next lemma, which establishes the desired estimates on solutions of (1.12) with a little extra regularity on φ , contains the heart of the proof.

Lemma 6. Let $\varphi \in P_0$, $u_0 \in L^1(\Omega) \cap L^\infty(\Omega)$ and $\varphi(u_0) \in L^1(\Omega)$. Let u be the solution of (1.12).

(i) Let $\varphi \in C^2(\mathbb{R} \setminus \{0\})$ and satisfy (1.4). Then

$$(1.16) \quad \|\frac{du}{dt}\|_1 \leq \frac{C(m, M)}{t} \|u_0\|_1$$

with $C(m, M) = 2(M+1)(m+2M)/m^2$.

(ii) Let $\varphi \in C^2(0, \infty)$, $\varphi/\varphi' \in C^1([0, \infty))$ and satisfy (1.8). Let $u_0 > 0$. Then

$$(1.17) \quad v \frac{du}{dt} \geq -\frac{1}{mt} \frac{\varphi(u)}{\varphi'(u)}.$$

Remarks: If (1.8) is satisfied with $v = 1$ (and hence $m < 1$) and if $\varphi^{-1}(0) = [0, r_0]$, then for $r > r_0$

$$\frac{\varphi(r)^{1-m}}{1-m} \leq \frac{\varphi'(r)}{\varphi(r)^m} (r-r_0) \quad \text{or} \quad \frac{\varphi(r)}{\varphi'(r)} \leq (1-m)(r-r_0).$$

If $v = -1$, then $r_0 = 0$ and $(\varphi/\varphi')(0+) = 0$, but $(\varphi/\varphi)'$ is not necessarily bounded in a neighborhood of 0. Because of this we impose the extra condition

$(\varphi/\varphi') \in C^1([0, \infty))$ in (ii). Note that the stronger condition (1.4) implies (1.6) and so

$$(v-m) > v(\frac{\varphi}{\varphi'})' > v-M, \quad |\frac{\varphi(r)}{\varphi'(r)}| < (M+1)|r|$$

on $R \setminus \{0\}$.

Proof of Lemma 5. Throughout the computations to follow we will use the fact that if

$p: R \rightarrow R$ is Lebesgue measurable and bounded, $j(r) = \int_0^r p(s)ds$ and $w \in W^{1,1}(0,T;L^1(\Omega))$, then $j(w) \in W^{1,1}(0,T;L^1(\Omega))$ and

$$\frac{d}{dt} j(w) = p(w) \frac{dw}{dt} \text{ a.e.}$$

In particular, the above relation with p equal to the characteristic function of a null set $N \subset R$ (so $j \equiv 0$) implies that $\frac{dw}{dt}(t,x) = 0$ a.e. on $\{(t,x): w(t,x) \in N\}$.

The above is well-known when $L^1(\Omega) = R$. For the reader's convenience a proof for this case is given in Lemma A.1 of the Appendix. The general case follows by use of Fubini's theorem.

The main part of the proof of the lemma is the introduction of the function

$$(1.18) \quad v = tu_t + \rho \frac{\varphi(u)}{\varphi'(u)},$$

where $\rho \in R$ is a parameter to be chosen, and the study of the equation satisfied by v . Here and below, the subscript t denotes differentiation in t .

It is first assumed that φ is locally Lipschitz on R for (i) and on $[0, \infty)$ for (ii). (This is implied by the assumptions if $v = 1$; if $v = -1$ we later approximate φ by φ_α .) Since $u \in W^{1,\infty}(0,T;L^1(\Omega))$, we have $\varphi(u) \in W^{1,1}(0,T;L^1(\Omega))$ and $\varphi(u)_t = \varphi'(u)u_t$. As B is linear and continuous (1.12) proves that $u \in W^{2,1}(0,T;L^1(\Omega))$ and

$$(1.19) \quad u_{tt} + B(\varphi'(u)u_t) = 0.$$

Differentiating (1.18) we find

$$(1.20) \quad v_t = u_t + tu_{tt} + \rho(\frac{\varphi}{\varphi'})'(u)u_t.$$

Taken together, (1.19) and (1.20) imply

$$(1.21) \quad \begin{cases} tv_t + B(t\varphi'(u)v) + G(u)v = \rho G(u) \frac{\varphi(u)}{\varphi'(u)} \\ \text{where } G(r) = -\rho(\frac{\varphi}{\varphi'})'(r) + \rho - 1 \end{cases}$$

Set

$$\text{sign } r = \begin{cases} \{1\} & r > 0 \\ [-1, 1] & r = 0 \\ \{-1\} & r < 0 \end{cases}, \quad \text{sign}^- r = \begin{cases} \{0\} & r > 0 \\ [0, -1] & r = 0 \\ \{-1\} & r < 0 \end{cases}.$$

A selection out of $\text{sign} v$ means a measurable function α such that

$\alpha(x) \in \text{sign} v(t, x)$ a.e. x , etc. To prove (i), multiply (1.21) by a selection out of $\text{sign} v$ (which is a subset of $\text{sign} \varphi'(u)v$) and use the accretivity of B in $L^1(\Omega)$ to conclude

$$(1.22) \quad t \frac{d}{dt} \int_{\Omega} |v| + \int_{\Omega} |G(u)| |v| < |\rho| \int_{\Omega} |G(u)| \frac{|\varphi(u)|}{\varphi'(u)}.$$

If we choose $\rho = 2v/m$ the assumptions on φ imply

$$(1.23) \quad 1 < G(r) < \frac{2M}{m} - 1, \quad \frac{|\varphi(r)|}{\varphi'(r)} < (M+1)|r|.$$

The estimates (1.23) used in (1.22) and integration in time of the result yields

$$(1.24) \quad t \int_{\Omega} |v| + \int_0^t \int_{\Omega} (G(u)-1) |v| < \frac{2}{m} \frac{2M}{m} (M+1) \int_0^t \int_{\Omega} |u| < \frac{4M(M+1)}{m^2} t \|u_0\|_1,$$

where the last inequality comes from the accretivity of $B\varphi$ in $L^1(\Omega)$ which implies

$\|u\|_1$ is nonincreasing. From (1.24) we have

$$\|v\|_1 < \frac{4M(M+1)}{m^2} \|u_0\|_1$$

and this with the definition (1.18) of v implies

$$t \|u_t\|_1 < \frac{4M(M+1)}{m^2} \|u_0\|_1 + \frac{2}{m} \int_{\Omega} \frac{|\varphi(u)|}{\varphi'(u)} < \frac{2(M+1)}{m} \left(1 + \frac{2M}{m}\right) \|u_0\|_1,$$

whence the result.

For (ii), we chose $\rho = v/m$ which implies $G(r) > 0$ on $(0, \infty)$. Then multiply (1.21) by a selection out of $v \operatorname{sign}^-(vv)$ (which is a subset of $v \operatorname{sign}^-(v\varphi'(u)v)$) and use (A3) for B to conclude

$$-t \frac{d}{dt} \int_{\Omega} (vv)^-(t) > 0.$$

Since $vv(0) = \frac{1}{m} \frac{\varphi(u_0)}{\varphi'(u_0)} > 0$, the above implies $vv(t) > 0$. Recalling the definition (1.18) of v this implies

$$(1.25) \quad vt u_t > -\frac{1}{m} \frac{\varphi(u)}{\varphi'(u)}.$$

This implies $vt\varphi(u)_t > -\varphi(u)/m$ which is equivalent to $(vt^{v/m}\varphi(u))_t > 0$.

When φ is not locally Lipschitz on \mathbb{R} (or $[0, \infty)$ for (ii)) we approximate φ by its Yosida approximation φ_α and u by the u_α of (1.11). Unfortunately, φ_α need not satisfy (1.4). Indeed, $\varphi_\alpha(r) = \varphi(\gamma_\alpha(r))$ where $\gamma_\alpha(r) = (I + \alpha\varphi)^{-1}(r)$ and so

$$\begin{cases} \varphi'_\alpha(r) = \varphi'(\gamma_\alpha(r))/(1 + \alpha\varphi'(\gamma_\alpha(r))) & \text{and} \\ \frac{d}{dr} \frac{\varphi_\alpha(r)}{\varphi'_\alpha(r)} = 1 - \frac{\varphi(\gamma_\alpha(r))\varphi''(\gamma_\alpha(r))}{[\varphi'(\gamma_\alpha(r))^2][1 + \alpha\varphi'(\gamma_\alpha(r))]} \end{cases}.$$

It follows that if (1.4) holds, G_α is defined as in (1.21) with φ_α in place of φ and $\rho = 2v/m$ then

$$(1.26) \quad \left| \frac{d}{dr} \frac{\varphi_\alpha}{\varphi'_\alpha} \right| \leq 1 + M \quad \text{and} \quad |G_\alpha(r)| \leq 2M/m.$$

Since φ_α is Lipschitz, computations leading to (1.22) are valid with $u_\alpha, v_\alpha = t u_{\alpha t} + \rho\varphi_\alpha(u_\alpha)/\varphi'_\alpha(u_\alpha)$, G_α in place of u, v, G and integration together with (1.26) gives

$$(1.27) \quad t \int_{\Omega} |v_\alpha| + \int_0^t \int_{\Omega} (G_\alpha(u_\alpha) - 1) |v_\alpha| \leq \frac{4M}{m^2} (1+M)t \|u_0\|_1.$$

By Lemma 4, $u_{\alpha t} + u_t$ in $L^1(0, T; L^1(\Omega))$. It also follows from (1.26) and $u_\alpha + u$ in $C([0, T]; L^1(\Omega))$ that $\varphi_\alpha(u_\alpha)/\varphi'_\alpha(u_\alpha) + \varphi(u)/\varphi'(u)$ in $L^1(0, T; L^1(\Omega))$. Hence $v_\alpha + v$ in $L^1(0, T; L^1(\Omega))$. Since $\varphi \in C^2(\mathbb{R} \setminus \{0\})$, $G_\alpha(r)$ converges to $G(r)$ for $r \neq 0$. Hence $G_\alpha(u_\alpha)|v_\alpha|$ (interpreted as 0 at points where u_α vanishes, since $v_\alpha = 0$ a.e. on $\{(t, x): u_\alpha(t, x) = 0\}$) converges to $G(u)|v|$ in $L^1(0, T; L^1(\Omega))$. Thus one may pass to the limit in (1.27) to obtain the desired conclusion.

To obtain (ii) we also approximate u by u_α as above. The estimates (1.26) need not hold now, but since $\varphi/\varphi' \in C^1([0, \infty))$, $(\varphi_\alpha/\varphi'_\alpha)'$ and G_α remain locally bounded on $[0, \infty)$ uniformly in α . Hence the convergence assertions above remain valid and we can pass to the limit in the inequality

$$-t \frac{d}{dt} \int_{\Omega} (vv_\alpha)^- - \int_{\Omega} G_\alpha(u_\alpha)(vv_\alpha)^- > -\frac{1}{m} \int_{\Omega} (G_\alpha(u_\alpha))^- \frac{\varphi_\alpha(u_\alpha)}{\varphi'_\alpha(u_\alpha)}$$

which is deduced from the α -version of (1.21) with $\rho = v/m$ as before. Since $G(u) > 0$, we then obtain that $\int_{\Omega} (vv)^-$ is nondecreasing and finish as before.

Proof of Theorem 3:

There are three steps of the proof remaining. We first show that if φ satisfies (1.3) and (1.4), then it can be locally approximated by functions φ_n satisfying the hypothesis of Lemma 5 in such a way that $e^{-tB\varphi_n}$ converges suitably to $e^{-tB\varphi}$. Then we show that $B\varphi = (\varepsilon I + A_\lambda)\varphi$ converges to $(\varepsilon I + A)\varphi$ as $\lambda \rightarrow 0$. Finally we deduce (1.5) as a consequence of (1.16) in the various limits.

We know that (1.4) may be restated as the Lipschitz continuity of φ/φ' (extended as zero on $\{\varphi=0\}$) together with

$$(1.28) \quad v-m > v(\frac{\varphi}{\varphi'})' > v-M.$$

Let $g \in C([0, \infty))$ be of locally bounded variation, $g(0) = 0$, and consider the approximations g_n , $n = 1, 2, \dots$ given by

$$(1.29) \quad g_n(r) = T_n g(r) = n \int_0^r e^{n(s-r)} g(s) ds = g(r) - \int_0^r e^{n(s-r)} dg(s)$$

so that

$$(1.30) \quad g'_n(r) = n \int_0^r e^{n(s-r)} dg(s).$$

From (1.29) we see that $T_n g$ is C^1 and converges as $n \rightarrow \infty$ uniformly to g on compact sets. Moreover, if g is nondecreasing, then $g_n(r)$ is nondecreasing in r as well as

n and g_n increases to g . Moreover, from (1.30) we see that $0 < g' \leq K$ implies

$0 < g'_n \leq K$. Set $g(r) = (v-m)r - v\varphi(r)/\varphi'(r)$ and define φ_n on $[0, \infty)$ by

$v\varphi_n/\varphi'_n = (v-m)r - T_n g$, $\varphi_n(r_1) = \varphi(r_1)$, where r_1 is chosen so that $\varphi(r_1) > 0$ and large enough for what comes later. Since

$$(1.31) \quad \varphi_n(r) = \varphi(r_1) \exp \int_{r_1}^r \frac{\varphi'_n(s)}{\varphi_n(s)} ds,$$

the above consideration imply that φ_n satisfies the assumptions of Lemma 5 (with the same M and m as φ) and φ_n decreases (respectively, increases) to φ on $[0, r_1]$ if $v = 1$ (respectively, $v = -1$). We could likewise arrange that φ_n converge monotonically to φ in the opposite sense by choosing $g(r) = (v-M)r - v\varphi(r)/\varphi'(r)$ (which is nonincreasing), $\varphi_n/\varphi'_n = (v-M)r - T_n g$. The analogous process is done on $(-\infty, 0]$ to define φ_n on \mathbb{R} so that φ_n converges monotonically to φ on $[-r_1, r_1]$.

Now, for $f \in L^1(\Omega)$ with $\|f\|_\infty \leq r_1$, let $\lambda > 0$ and u_n be the solution of $u_n + \lambda B\varphi_n(u_n) = f$. By Proposition 1(v) applied to B , $\|u_n\|_p \leq \|f\|_p$ for $p = 1, \infty$. Hence u_n is bounded in $L^1(\Omega)$ and has its values in the interval for which $n + \varphi_n(r)$ is monotone. Since B^{-1} is bounded, $\varphi_n(u_n) = \lambda^{-1} B^{-1}(f - u_n)$ is also bounded in $L^1(\Omega)$. Since $n + \varphi_n(r)$ is monotone so is $n + \varphi_n(u_n)$ (Lemma a.2 of the Appendix). Hence $\varphi_n(u_n)$ converges in $L^1(\Omega)$ and so does u_n by continuity of B . The limit u clearly satisfies $u + \lambda B\varphi(u) = f$. It follows from the convergence theorem that $e^{-tB\varphi} u_0 + e^{-tB\varphi} u_0$ whenever $u_0 \in L^1(\Omega)$, $\|u_0\|_\infty \leq r_1$.

For the second step, let u_λ, u solve

$$u_\lambda + (\epsilon I + A_\lambda)\varphi(u_\lambda) = f, \quad u + (\epsilon I + A)\varphi(u) = f,$$

respectively, where $f \in L^1(\Omega)$ and $\varphi \in P_0$. Rewriting the second equation as

$u + (\epsilon I + A_\lambda)\varphi(u) = f + A_\lambda\varphi(u) - A\varphi(u)$ and using this, the first equation and accretivity we find

$$\|u - u_\lambda\|_1 \leq \|A_\lambda\varphi(u) - A\varphi(u)\|_1.$$

Since A is linear, densely defined and m -accretive, $A_\lambda v \rightarrow Av$ as $\lambda \rightarrow 0$ for all $v \in D(A)$ and we conclude that $u_\lambda \rightarrow u$ in $L^1(\Omega)$. Thus $(\epsilon I + A_\lambda)\varphi \rightarrow (\epsilon I + A)\varphi$.

Now let φ satisfy (1.3), (1.4) and $u_0 \in \overline{D((\epsilon I + A)\varphi)}$. Choose $u_0^j \in L^1(\Omega) \cap L^\infty(\Omega)$ whose support is of finite measure so that $u_0^j \rightarrow u_0$ in $L^1(\Omega)$. With φ_n as above, we fix j and let $r_1 > \|u_0^j\|_\infty$. Set

$$u_n = e^{-tB\psi} u_0^j.$$

By Lemma 6

$$\left\| \frac{du_n}{dt} \right\|_1 \leq \frac{C(m, M)}{t} \|u_0^j\|_1.$$

Since $e^{tB\psi} u_0^j + e^{-tB\psi} u_0^j$ uniformly for bounded t and $u^j(t) = e^{-tB\psi} u_0^j \in W^{1,1}([0, \infty); L^1(\Omega))$ by Lemma 5, the above inequality is correct with u^j in place of u_n . Moreover, since $t \rightarrow \left\| \frac{du^j}{dt}(t) \right\|_1$ is nonincreasing, we have

$$(1.31) \quad \|u^j(t+h) - u^j(t)\|_1 \leq \frac{h}{t} C(m, M) \|u_0^j\|_1.$$

Now $e^{-tB\psi} u_0^j \rightarrow e^{-tB\psi} u_0$ as $j \rightarrow \infty$, so (1.31) holds with $u_\varepsilon = e^{-t(\varepsilon I + A)\psi} u_0$ in place of u^j . We may then send ε to 0 to find (1.5).

Proof of Theorem 4.

The property (1.9) can be obtained from Lemma 6(ii) by successive approximations as above. The assumption (1.8) implies that $(v-m)r - \psi(r)/\psi'(r)$ is nondecreasing. As above, it is the increasing limit of C^1 nondecreasing functions on $[0, \infty]$ and we can construct ψ_n satisfying the assumptions of Lemma 6(ii) converging monotonically to ψ on any $[0, r_1]$. The rest is as above.

For (1.10), if u is a solution of (1.12), we use that

$$\int_{\Omega} |u_\varepsilon| = \int_{\Omega} (u_\varepsilon + 2(u_\varepsilon)^-)$$

by (A.3) applied to B , $\int_{\Omega} u_\varepsilon \leq 0$. By (1.17)

$$\int_{\Omega} u_\varepsilon^- \leq \left(\frac{1}{m} - 1\right) \frac{1}{\varepsilon} \int_{\Omega} u \leq \left(\frac{1-m}{m}\right) \frac{1}{\varepsilon} \int_{\Omega} u_0.$$

Hence $\int_{\Omega} |u_\varepsilon| \leq 2 \frac{1-m}{m\varepsilon} \|u_0\|_1$, and (1.10) follows.

Appendix

Proof of Proposition 2.

Let $\varphi \in P_0$ and A satisfy the hypotheses of Proposition 2. We may simply define A_φ by

$$(a.1) \quad \begin{aligned} g \in A_\varphi(u) \text{ if } \exists \lambda > 0 \text{ and } f \in L^1(\Omega) \text{ such} \\ \text{that if } u_\varepsilon = (I + \lambda((\varepsilon I + A)\varphi))^{-1}f \text{ then} \\ \lim_{\varepsilon \rightarrow 0} u_\varepsilon = u \text{ and } g = \lambda^{-1}(f - u). \end{aligned}$$

To see that A_φ extends $A\varphi$, observe that if $u \in D(A\varphi)$, $f = u + \lambda A\varphi(u)$, and $u_\varepsilon = (I + \lambda(\varepsilon I + A)\varphi)^{-1}f$ then

$$\begin{aligned} \|u_\varepsilon - u\|_{L^1} &= \|(I + \lambda(\varepsilon I + A)\varphi)^{-1}f - (I + \lambda(\varepsilon I + A)\varphi)^{-1}(f + \varepsilon \varphi(u))\|_{L^1} \\ &\leq \varepsilon \|\varphi(u)\|_{L^1} \rightarrow 0 \end{aligned}$$

so $\lambda^{-1}(f - u_\varepsilon) \rightarrow \lambda^{-1}(f - u) = A\varphi(u)$. To see that $A\varphi$ is accretive, set $C_\varepsilon = (\varepsilon I + A)\varphi$ for $\varepsilon > 0$. Now if $g \in A_\varphi u$, (a.1) implies the existence of u_ε such that $u_\varepsilon \rightarrow u$ and $C_\varepsilon u_\varepsilon \rightarrow g$, i.e. $A_\varphi \subset \liminf_{\varepsilon \rightarrow 0} C_\varepsilon$. But the limit inferior of a family of accretive operators is clearly accretive.

We next show $R(I + \lambda A\varphi) \supset L^1(\Omega)^+$, $(I + \lambda A_\varphi)^{-1}L^1(\Omega)^+ \subset L^1(\Omega)^+$ and $(I + \lambda(\varepsilon I + A)\varphi)^{-1}f \rightarrow (I + \lambda A\varphi)^{-1}f$ for $\lambda > 0$ and $f \in L^1(\Omega)^+$. This merely requires showing that if $\lambda > 0$, $f \in L^1(\Omega)^+$ and u_ε solves

$$(a.2) \quad u_\varepsilon + \lambda(\varepsilon \varphi(u_\varepsilon) + A\varphi(u_\varepsilon)) = f$$

then $u_\varepsilon \geq 0$ and $\lim_{\varepsilon \rightarrow 0} u_\varepsilon$ exists. Now $u_\varepsilon \geq 0$ follows from Proposition 1, as does the estimate

$$(a.3) \quad \|u_\varepsilon + \lambda \varepsilon \varphi(u_\varepsilon)\|_1 = \|u_\varepsilon\|_1 + \lambda \varepsilon \|\varphi(u_\varepsilon)\|_1 \leq \|f\|_1.$$

Moreover, we show that u_ε is nonincreasing in ε . This monotonicity and the estimate (a.3) imply $\lim_{\varepsilon \rightarrow 0} u_\varepsilon$ exists. Indeed, if $\varepsilon > \eta > 0$ we have (because $\varphi(u_\varepsilon) > 0$)

$$u_\varepsilon + \lambda(\eta\varphi(u_\varepsilon) + A\varphi(u_\varepsilon)) = f - \lambda(\varepsilon - \eta)\varphi(u_\varepsilon) < f$$

$$u_\eta + \lambda(\eta\varphi(u_\eta) + A\varphi(u_\eta)) = f.$$

Now by Proposition 1, $(I + \lambda(\eta I + A)\varphi)^{-1}$ is order preserving and thus

$$u_\varepsilon < u_\eta.$$

Remark: It would be nice (especially below) if $\varepsilon\varphi(u_\varepsilon) \rightarrow 0$ in $L^1(\Omega)$, in which case the current task would be quite simple. However, examples show this to be false in general.

The final assertion of Proposition 2 has already been verified. We consider next the case in which $|\varphi(x)| < K|x|$ on $|x| < r_0$. We now seek to show that if u_ε solves (a.2) and $f \in L^1(\Omega)$ is arbitrary, then $\lim_{\varepsilon \rightarrow 0} u_\varepsilon$ exists. Since $(I + \lambda(\varepsilon I + A)\varphi)^{-1}$ is nonexpansive, it suffices to choose f from the dense set $L^1(\Omega) \cap L^\infty(\Omega)$. Then $\|u_\varepsilon\|_\infty < \|f\|_\infty$ and there is another constant K_1 such that $|\varphi(x)| < K_1|x|$ on $|x| < \|f\|_\infty$. Hence $\varphi(u_\varepsilon)$ is bounded in $L^1(\Omega)$ by $K_1\|u_\varepsilon\|_1$. Since (a.3) still holds, $\varepsilon\varphi(u_\varepsilon) \rightarrow 0$ in $L^1(\Omega)$ and

$$\begin{aligned} \|u_\varepsilon - u_\eta\|_1 &= \|(I + \lambda A\varphi)^{-1}(f - \lambda\varepsilon\varphi(u_\varepsilon)) - (I + \lambda A\varphi)^{-1}(f - \lambda\eta\varphi(u_\eta))\|_1 \\ &< \lambda(\|\varepsilon\varphi(u_\varepsilon)\|_1 + \|\eta\varphi(u_\eta)\|_1) \end{aligned}$$

so u_ε is Cauchy in $L^1(\Omega)$ as $\varepsilon \rightarrow 0$. The case $\text{meas}\Omega < \infty$ is similar, since then $\varepsilon\varphi(u_\varepsilon) \rightarrow 0$ in $L^\infty(\Omega)$ implies the convergence in $L^1(\Omega)$.

Remark: The above proof shows that A_φ is the closure of $A\varphi$ in these cases. With $\Omega = \mathbb{R}$, $\varphi(x) = x^3$, $A = 0$ we have an example where $A_\varphi \neq A\varphi$.

The remaining case is the one in which φ is strictly monotone. Again let $f \in L^1(\Omega) \cap L^\infty(\Omega)$ and u_ε solve (a.2). Let $u_{\varepsilon+}$, $u_{\varepsilon-}$ solve

$$u_{\varepsilon v} + \lambda(\varepsilon \varphi(u_{\varepsilon v}) + \lambda \varphi(u_{\varepsilon v})) = v f^v, \quad v = \pm.$$

By the order preserving properties, $u_{\varepsilon-} < u_\varepsilon < u_{\varepsilon+}$. Moreover, by the first case treated above, $u_{\varepsilon v}$ converges monotonically as $\varepsilon \rightarrow 0$ to $u_v \in L^1(\Omega)$ and so $u_- < u_\varepsilon < u_+$. Therefore, by the dominated convergence theorem, it is enough to show that for $\delta > 0$

$$\lim_{\varepsilon, \eta \rightarrow 0} \text{meas}\{|u_\varepsilon - u_\eta| > \delta\} = 0.$$

Since u_ε, u_η are bounded and φ is strictly monotone, there is a $\mu > 0$ such that $\{|u_\varepsilon - u_\eta| > \delta\} \subset \{|\varphi(u_\varepsilon) - \varphi(u_\eta)| > \mu\}$. Now

$$(u_\varepsilon - u_\eta) + \lambda A(\varphi(u_\varepsilon) - \varphi(u_\eta)) = (\eta \varphi(u_\eta) - \varepsilon \varphi(u_\varepsilon)).$$

Let $p(r) = 1$ if $r > \mu$, $p(r) = -1$ if $r < -\mu$ and $p(r) = 0$ if

$|r| \leq \mu$. Multiply the above by $p(\varphi(u_\varepsilon) - \varphi(u_\eta))$, integrate, and use

(A.3) to conclude

$$\int_{\{|\varphi(u_\varepsilon) - \varphi(u_\eta)| > \mu\}} |u_\varepsilon - u_\eta| \leq \int_{\{|\varphi(u_\varepsilon) - \varphi(u_\eta)| > \mu\}} (|\eta \varphi(u_\eta) - \varepsilon \varphi(u_\varepsilon)|)$$

Now let $\kappa > 0$ be such that $\{|u_\varepsilon - u_\eta| > \kappa\} \subset \{|\varphi(u_\varepsilon) - \varphi(u_\eta)| > \mu\}$. There is such a κ because φ is continuous. We have, by the above,

$$\kappa \text{meas}\{|\varphi(u_\varepsilon) - \varphi(u_\eta)| > \mu\} \leq \|\eta \varphi(u_\eta) - \varepsilon \varphi(u_\varepsilon)\|_2 \text{meas}\{|\varphi(u_\varepsilon) - \varphi(u_\eta)| > \mu\}^{1/2}.$$

But $\varepsilon \varphi(u_\varepsilon)$ is bounded in $L^1(\Omega)$ and tends to zero in $L^\infty(\Omega)$. Thus

$\varepsilon \varphi(u_\varepsilon) \rightarrow 0$ in $L^p(\Omega)$, $1 < p < \infty$ and we conclude that

$\text{meas}\{|\varphi(u_\varepsilon) - \varphi(u_\eta)| > \mu\}$ and so $\text{meas}\{|u_\varepsilon - u_\eta| > \delta\}$ tends to zero as

$\varepsilon, \eta \rightarrow 0$, thus completing the proof.

Remarks:

(a) We do not know if $(I + \lambda(\varepsilon I + A)\varphi)^{-1}$ converges as $\varepsilon \rightarrow 0$ for every $\varphi \in P_0$.

(8) The definition of A_φ is consistent with known examples. One important case is $\Omega = \mathbb{R}^N$ and $A = -\Delta$. The construction of [3] coincides with ours when Proposition 2 applies, however in [3] precise information on the domain of A_φ is obtained and more general φ 's are permitted.

Lemma a.1. Let $p: \mathbb{R} \rightarrow \mathbb{R}$ be Lebesgue measurable, bounded and $j(x) = \int_0^x p(s)ds$. Let $w \in W^{1,1}(0,T; L^1(\Omega))$. Then $j(w) \in W^{1,1}(0,T; L^1(\Omega))$ and

$$(a.4) \quad \frac{d}{dt} j(w) = p(w) \frac{dw}{dt} \quad \text{a.e.}$$

Proof. Let us treat the case $L^1(\Omega) = \mathbb{R}$. Then the general case follows by using Fabini's theorem and looking directly at $\lim_{h \rightarrow 0} (j(w(t+h)) - j(w(t)))/h$.

One has to prove

$$(a.5) \quad -\int_0^T \psi'(t) j(w(t)) dt = \int_0^T \psi(t) p(w(t)) w'(t) dt \quad \forall \psi \in C_0^\infty(0,T)$$

with the proof demonstrating the measurability of $p(w)w'$ so that the equation has a meaning. Notice that if (a.4) holds for a sequence (p_n, j_n) in place of (p, j) and p_n converges boundedly everywhere to p , then (a.5) holds for (p, j) .

The relation (a.5) is obvious if p is continuous. If O is open in \mathbb{R} and p is the characteristic function χ_O of O , then p is the increasing limit of continuous functions. Hence (a.5) holds with $p = \chi_O$.

If $N \subset \mathbb{R}$ is a null set, then there is a decreasing sequence O_n of open sets such that $O_n \supset N$ and $\text{meas } O_n \rightarrow 0$. Let $N' = \bigcap_n O_n$ so that $\chi_{N'} = \lim_{n \rightarrow \infty} \chi_{O_n}$ is the decreasing limit of characteristic functions of open sets. By the above remarks, (a.5) holds with $j = 0$ and $p = \chi_{N'}$, so $0 = \chi_{N'}(w)w'$ a.e. and $w' = 0$ a.e. on $\{t \in (0,T): w(t) \in N \subset N'\}$.

If $E \subset (0,T)$ is measurable, then there exists a decreasing sequence O_n of open sets such that $O_n \supset E$ and $\text{meas}(\bigcap_n O_n \setminus E) = 0$. Set $E' = \bigcap_n O_n$. We have, by the above remarks, $\chi_{E'}(w)w' = \chi_E(w)w' + \chi_{E' \setminus E}(w)w'$ a.e. and the validity of (a.5) for $p = \chi_E$, implies

the validity for χ_E . Since any bounded measurable function is the uniform limit of a sequence of simple functions the proof is complete.

Lemma a.2. Let B be linear, densely defined and satisfy (A.1), (A.2). Let $\varphi, \psi \in P_0$ and $\varphi(r) > \psi(r)$ for all r . Let $u \in D(B\varphi)$, $v \in D(B\psi)$ and

$$u + B\varphi(u) = v + B\psi(v).$$

Then $\varphi(u) > \psi(v)$.

Proof. We have

$$(a.6) \quad u - v + B(\varphi(u) - \psi(v)) = 0.$$

Set

$$p(x) = \begin{cases} -1 & \text{if } \varphi(u(x)) < \psi(v(x)) \\ 0 & \text{otherwise.} \end{cases}$$

Then $p(x) \in \beta(\varphi(u(x)) - \psi(v(x)))$ where

$$\beta(r) = \begin{cases} \{0\} & \text{if } r > 0 \\ [0, 1] & \text{if } r = 0 \\ \{0\} & \text{if } r < 0. \end{cases}$$

Moreover, $(u - v)p(\varphi(u) - \psi(v)) = |u - v|$ on $\{\varphi(u) < \psi(v)\}$ by the monotonicity of φ and $\varphi > \psi$, while $(u - v)p(\varphi(u) - \psi(v)) > 0$.

Multiplying (a.6) by $p(\varphi(u) - \psi(v))$ and integration with the use of (A.3) yields

$$\int_{\{\varphi(u) < \psi(v)\}} |u - v| < 0$$

so $\varphi(u) > \psi(v)$.

Remark: If it is known that $\|u\|_\infty, \|v\|_\infty < r_1$ and $\varphi(r) > \psi(r)$ holds for $|r| < r_1$ we clearly have the same conclusion.

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MGC:MP/db

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2187 ✓	2. GOVT ACCESSION NO. AD-A099349	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) REGULARIZING EFFECTS FOR $u_t + A\varphi(u) = 0$ IN L^1 .		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Michael G. Crandall and Michel Pierre		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 MCS-7927062
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18		12. REPORT DATE March 1981
		13. NUMBER OF PAGES 21
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES U. S. Army Research Office P. O. Box 12211 Research Triangle Park North Carolina 27709 National Science Foundation Washington, D. C. 20550		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) regularizing effect, porous flow equations, conservation laws, nonlinear semigroups		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Various initial-boundary value problems and Cauchy problems can be written in the form $\frac{du}{dt} + A\varphi(u) = 0$ where $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing and A is the linear generator of strongly continuous nonexpansive semigroup e^{-tA} in an L^1 space. For example, if $A = -\Delta$ (subject, perhaps, to suitable boundary conditions) we obtain equations arising in flow in a porous medium or plasma physics (depending on the choice of φ) while if $A = \frac{\partial}{\partial x}$ acting in $L^1(\mathbb{R})$ we have a scalar conservation law. In this paper we show that if $M, m > 0$ and $m\varphi'^2 \leq \varphi\varphi'' \leq M\varphi'^2$, where $\varphi \in \{1, -1\}$, then (roughly speaking), the norm of tdu/dt may be estimated in terms of the		

20. Abstract (continued)

initial data u_0 in L^1 . Such estimates give information about the regularity of solutions, asymptotic behaviour, etc., in applications.

Side issues, such as the introduction of sufficiently regular approximate problems on which estimates can be made and the assignment of a precise meaning to the operator " A_ϵ ", are also dealt with. These considerations are of independent interest.

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